Solving The Black-Scholes Option Pricing Integral

Gary Schurman, MBE, CFA

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We will define the variable z to be a normally-distributed random variable with mean zero and variance one. This statement in equation form is...

$$z \sim N\Big[0,1\Big] \tag{1}$$

We will define the variable S_t to be stock price at time t, the variable C_t to be call option price at time t, the variable α to be the risk-free rate, the variable ϕ to be the dividend yield, the variable σ to be stock return volatility, the variable X to be call option exercise price, and the variable T to be time until option expiration (in years). Using Equation (1) above the equation for the Black-Scholes Option Pricing Model in integral form is...

$$C_0 = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\mathrm{Max}\bigg[S_0\,\mathrm{Exp}\bigg\{\bigg(\alpha-\phi-\frac{1}{2}\,\sigma^2\bigg)T+\sigma\,\sqrt{T}\,z\bigg\}-X,0\bigg]\,\mathrm{Exp}\bigg\{-\alpha\,T\bigg\}\,\delta z \tag{2}$$

In this white paper we will solve the integral in Equation (2) above.

The Normal Distribution

The equation for the cumulative normal distribution function is...

$$\text{Area under the normal curve} = \int\limits_{-\infty}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\delta z = 1.00 \;\;\text{...where...}\;\; z \sim N\Big[0,1\Big] \tag{3}$$

Using Equation (3) above the probability that the random variable x (distributed normally with mean zero and variance one) is greater than some value a is...

$$\mathrm{Prob}\Big[z > a\Big] = \int\limits_{a}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\delta z = 1-\text{NORM.S.DIST(a,TRUE)} \tag{4}$$

Using Equation (4) above we will define the standardized cumulative normal distribution function (CNDF) as follows...

$$\mathrm{CNDF}(a) = \int\limits_{a}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\delta z \;\;\text{...where...}\;\; z \sim N\Big[0,1\Big] \tag{5}$$

Note that because the normal distribution is symmetrical we can rewrite Equation (5) above as...

$$\mathrm{CNDF}(a) = \int\limits_{-\infty}^{-a} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,z^2\bigg\}\,\delta z = \text{NORM.S.DIST(-a,TRUE)} \tag{6}$$

The Equation For Stock Price

In Equation (2) above we defined the equation for stock price at time T (option expiration) under the risk-neutral Measure Q as follows...

$$S_T = S_0\,\mathrm{Exp}\bigg\{\bigg(\alpha-\phi-\frac{1}{2}\,\sigma^2\bigg)T+\sigma\,\sqrt{T}\,z\bigg\} \;\;\text{...where...}\;\; z \sim N\Big[0,1\Big] \tag{7}$$

Under the risk-neutral measure all assets earn a total rate of return (capital gains plus dividends) equal to the risk-free rate, which means that the stock will record expected capital gains equal to the risk-free rate minus the dividend yield. Using Equations (3) and (7) above the equation for expected stock price at time T under the risk-neutral Measure Q is...

$$\mathbb{E}^Q\left[S_T\right]=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,z^2\right\}S_0\,\mathrm{Exp}\left\{\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,z\right\}\delta z=S_0\,\mathrm{Exp}\left\{(\alpha-\phi)\,T\right\} \tag{8}$$

The Black-Scholes Option Pricing Integral

We want to remove the Max function from Equation (2) above. We do this by defining the variable a to be the value of the random variable z such that stock price at time T equals the option exercise price (i.e. the option is at-the-money). Using Equation (7) above we want the following equation to hold...

$$S_0\,\mathrm{Exp}\left\{\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,a\right\}=X \tag{9}$$

Using Equation (9) above and solving for the variable a we get the following equation...

$$\begin{aligned} S_0\,\mathrm{Exp}\left\{\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,a\right\}&=X\\ \left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,a&=\ln\left(\frac{X}{S_0}\right)\\ \sigma\,\sqrt{T}\,a&=\ln\left(\frac{X}{S_0}\right)-\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\\ a&=\left[\ln\left(\frac{X}{S_0}\right)-\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T} \end{aligned} \tag{10}$$

Using Equation (10) above we can remove the Max function and rewrite Equation (2) above as...

$$C_0=\int\limits_{a}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,z^2\right\}\left[S_0\,\mathrm{Exp}\left\{\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,z\right\}-X\right]\mathrm{Exp}\left\{-\,\alpha\,T\right\}\delta z \tag{11}$$

We will define integral one as follows...

$$I_1=\int\limits_{a}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,z^2\right\}S_0\,\mathrm{Exp}\left\{\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,z\right\}\mathrm{Exp}\left\{-\,\alpha\,T\right\}\delta z \tag{12}$$

We will define integral two as follows...

$$I_2=\int\limits_{a}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,z^2\right\}X\,\mathrm{Exp}\left\{-\,\alpha\,T\right\}\delta z \tag{13}$$

Using Equations (12) and (13) above we can rewrite Equation (11) above as...

$$C_0=I_1-I_2 \tag{14}$$

The Solution To Integral One

Note that we can rewrite Equation (12) above as...

$$I_1 = S_0 \,\mathrm{Exp}\left\{\left(\alpha - \phi - \frac{1}{2}\,\sigma^2\right)T\right\} \mathrm{Exp}\left\{-\,\alpha\,T\right\} \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\,z^2\right\} \mathrm{Exp}\left\{\sigma\,\sqrt{T}\,z\right\} \delta z$$

$$= S_0 \,\mathrm{Exp}\left\{-\,\phi\,T - \frac{1}{2}\,\sigma^2\,T\right\} \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\left(z^2 - 2\,\sigma\,\sqrt{T}\,z\right)\right\} \delta z$$

$$= S_0 \,\mathrm{Exp}\left\{-\,\phi\,T - \frac{1}{2}\,\sigma^2\,T\right\} \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\left(z^2 - 2\,\sigma\,\sqrt{T}\,z + \sigma^2 T\right)\right\} \mathrm{Exp}\left\{\frac{1}{2}\,\sigma^2 T\right\} \delta z$$

$$= S_0 \,\mathrm{Exp}\left\{-\,\phi\,T\right\} \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\left(z^2 - 2\,\sigma\,\sqrt{T}\,z + \sigma^2 T\right)\right\} \delta z \tag{15}$$

We will define the variable θ as follows...

$$\theta = z - \sigma\,\sqrt{T} \;\text{ ...where... }\; \theta^2 = z^2 - 2\,\sigma\,\sqrt{T}\,z + \sigma^2\sqrt{T} \;\text{ ...and... }\; \frac{\delta\theta}{\delta z} = 1 \;\text{ ...such that... }\; \delta z = \delta\theta \tag{16}$$

Using Equations (5) and (16) above we can rewrite Equation (15) above as...

$$I_1 = S_0 \,\mathrm{Exp}\left\{-\,\phi\,T\right\} \int\limits_{a-\sigma\,\sqrt{T}}^{\infty-\sigma\,\sqrt{T}} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\,\theta^2\right\} \delta\theta$$

$$= S_0 \,\mathrm{Exp}\left\{-\,\phi\,T\right\} \int\limits_{a-\sigma\,\sqrt{T}}^{\infty} \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\,\theta^2\right\} \delta\theta$$

$$= S_0 \,\mathrm{Exp}\left\{-\,\phi\,T\right\}\left(1 - \mathrm{CNDF}\left[a - \sigma\,\sqrt{T}\right]\right) \tag{17}$$

Using Equation (6) above we can rewrite Equation (17) above as...

$$I_1 = S_0 \,\mathrm{Exp}\left\{-\,\phi\,T\right\} \mathrm{CNDF}\left[-\,a + \sigma\,\sqrt{T}\right] \tag{18}$$

The Solution To Integral Two

Note that we can rewrite Equation (13) above as...

$$I_2 = \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\,z^2\right\} X\,\mathrm{Exp}\left\{-\,\alpha\,T\right\} \delta z$$

$$= X\,\mathrm{Exp}\left\{-\,\alpha\,T\right\} \int\limits_a^\infty \sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\,\frac{1}{2}\,z^2\right\} \delta z \tag{19}$$

Using Equation (5) above we can rewrite Equation (19) above as...

$$I_2 = X\,\mathrm{Exp}\left\{-\,\alpha\,T\right\}\left(1 - \mathrm{CNDF}(a)\right) \tag{20}$$

Using Equation (6) above we can rewrite Equation (20) above as...

$$I_2 = X\,\mathrm{Exp}\left\{-\,\alpha\,T\right\} \mathrm{CNDF}\left[-\,a\right] \tag{21}$$

The Black-Scholes Option Pricing Model Equation

The negative of the variable a as defined by Equation (10) above is...

$$-a=-\left[\ln\left(\frac{X}{S_0}\right)-\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T}=\left[\ln\left(\frac{S_0}{X}\right)+\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T} \tag{22}$$

Using Equation (22) above we will make the following definition...

$$\begin{aligned} d_1&=-a+\sigma\,\sqrt{T}\\ &=\left[\ln\left(\frac{S_0}{X}\right)+\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T}+\sigma\,\sqrt{T}\\ &=\left[\ln\left(\frac{S_0}{X}\right)+\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T+\sigma^2 T\right]\Big/\sigma\,\sqrt{T}\\ &=\left[\ln\left(\frac{S_0}{X}\right)+\left(\alpha-\phi+\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T} \end{aligned} \tag{23}$$

Using Equation (22) above we will make the following definition...

$$\begin{aligned} d_2&=-a\\ &=\left[\ln\left(\frac{S_0}{X}\right)+\left(\alpha-\phi-\frac{1}{2}\,\sigma^2\right)T\right]\Big/\sigma\,\sqrt{T} \end{aligned} \tag{24}$$

Using Equations (14), (18), (21), (23) and (24) above the solution to the Black-Scholes Option Pricing Model integral as defined by Equation (2) above is...

$$\begin{aligned} C_0&=\int\limits_{-\infty}^{\infty}\sqrt{\frac{1}{2\,\pi}}\,\mathrm{Exp}\left\{-\frac{1}{2}\,z^2\right\}\mathrm{Max}\left[S_0\,\mathrm{Exp}\left\{\left(\alpha-\frac{1}{2}\,\sigma^2\right)T+\sigma\,\sqrt{T}\,z\right\}-X,0\right]\mathrm{Exp}\left\{-\alpha\,T\right\}\delta z\\ &=S_0\,\mathrm{Exp}\left\{-\phi\,T\right\}\mathrm{CNDF}\left[d_1\right]-X\,\mathrm{Exp}\left\{-\alpha\,T\right\}\mathrm{CNDF}\left[d_2\right] \end{aligned} \tag{25}$$